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Shift and Stability of Ground States of a Nonlinear Schrödinger Equation Outside a Small Insulated Domain

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We study the behavior and stability of ground states of a Neumann problem in the exterior of a shrinking domain. We show that the ground states concentrate at a finite point which is a global maximum point of some function related to the domain. Then we use this result to prove the stability of the corresponding solitary waves of the time-dependent nonlinear Schrödinger equation. © 1999 Academic Press

Key Words: ground states; solitary waves; stability; energy; nonlinear Schrödinger equations; exterior domains.

1. INTRODUCTION

The purpose of this paper is to study the behavior and stability of ground states of the following Neumann problem in the exterior of a shrinking domain

$$\begin{cases} \Delta u - u + u^p = 0, & u > 0 \quad \text{in} \quad \Omega_\varepsilon \subset R^n \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.1}$$

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where $1<p<(n+2)/(n-2)$, $n\geqslant 3$, $\varepsilon>0$, $\Omega_\varepsilon = R^n\backslash(\varepsilon\Omega)$, Ω is any fixed bounded smooth domain containing 0 in its interior with its complement having no bounded components, ν is the unit normal of $\partial\Omega_\varepsilon$, pointing away from Ω_ε.

It is standard that any non-zero constant multiple of a minimizer of the "*energy*" functional

$$I_\varepsilon(u)=\frac{\int_{\Omega_\varepsilon}(|\nabla u|^2+u^2)}{(\int_{\Omega_\varepsilon}|u|^{p+1})^{2/(p+1)}}, \qquad u\in H^1(\Omega_\varepsilon), \quad u\not\equiv 0 \tag{1.2}$$

is still a minimizer, that a minimizer never changes its sign, and that after multiplying it by a suitable constant the resulting minimizer is a solution of (1.1). Such solutions of (1.1) are called *ground states* and are always denoted by u_ε in this paper.

Esteban in [1] has shown, among other things that (i) for every $\varepsilon>0$, (1.1) always has a ground state u_ε; (ii) if Ω is the unit ball, then as ε shrinks to zero, there exists $\bar{x}_\varepsilon\in R^n$ such that

$$\|u_\varepsilon - w(\cdot-\bar{x}_\varepsilon)\|_{H^1(\Omega_\varepsilon)}\to 0, \tag{1.3}$$

where w is the unique solution of

$$\begin{cases} \Delta w - w + w^p = 0, & w>0 \quad \text{in } R^n, \\ w(0)=\max\limits_{x\in R^n} w(x), & w(x)\to 0 \quad \text{as} \quad |x|\to\infty. \end{cases} \tag{1.4}$$

(By [3], w is spherically symmetric about the origin and is radially decreasing, namely, $w(x)=w(r)$, $r=|x|$ with $w'(r)<0$ for $r>0$; by [5], w is unique.) Thus as the domain shrinks, u_ε vaguely resembles w with a shift. Esteban [1] posed an open problem asking if the shift is unbounded, or bounded or converging to zero as ε shrinks to 0. (In the last scenerio, u_ε would be a small perturbation of w without a shift. Then u_ε would be "almost radial" because w is radial.) If one looks only at the Neumann boundary condition in (1.1), one might be led to concluding that the first or the last should occur. However, as we are going to show (see Theorem 3.1), the second scenery occurs: As ε shrinks, u_ε converges (after passing to a subsequence) to w shifted to a point x_0 where x_0 is a global maximum point of the function

$$F_\Omega(x)=c_\Omega(x)\,\frac{|w'|^2(|x|)}{|x|^2}+w^2(|x|)-\frac{2}{p+1}\,w^{p+1}(|x|), \qquad x\in R^n, \tag{1.5}$$

where $c_\Omega(x)$ is a positive function depending on Ω, only to be defined later in Section 3 and when Ω is the unit ball, $c_\Omega(x)=n\,|x|^2/(n-1)$. (Note that

$0<|x_0|<\infty$, see Section 3.) Thus for ε small u_ε peaks at a point where F_Ω attains its global maximum. We emphasize that our result holds for any fixed bounded smooth domain Ω containing 0 in its interior with its complement having no bounded components.

The basic idea in proving this result is to first obtain a good lower bound of the "*ground energy*" $I_\varepsilon(u_\varepsilon)$, then to argue that if u_ε does not converge to w shifted to a maximum point of $F_\Omega(x)$, then $I_\varepsilon(u_\varepsilon)$ would be larger than the energy of a carefully constructed test function, and hence a contradiction would be reached. This is directly inspired by the work of Ni and Takagi [8] and [9].

This result can be used to solve an open problem in [2], where Esteban and Strauss considered the following nonlinear Schrödinger equation with Neumann boundary condition

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, & x \in \Omega_\varepsilon \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.6}$$

where $\Omega_\varepsilon = R^n \backslash (\varepsilon\Omega)$, Ω is the unit ball in R^n. (This equation has attracted a lot of attentions in recent years, see the references in [2].) The ground state u_ε of (1.1) can be used to construct a solitary wave solution of (1.6): $u(x,t) = e^{it}u_\varepsilon(x)$. It is shown in [2], among other things, that the solitary wave $e^{it}u_\varepsilon(x)$ is orbitally unstable if $(n+4)/n < p < (n+2)/(n-2)$ and if ε is large or small, and it is orbitally stable if $1 < p < (n+4)/n$ and ε is large. In the framework of [2], we shall use our result on the location of the peak of u_ε to show that if ε is small and $1<p<(n+4)/n$, the solitary wave $e^{it}u_\varepsilon(x)$ is orbitally stable. See Theorem 4.1.

For behavior of ground states u_ε when ε is large, we mention a result of Esteban [1] which says that as $\varepsilon \to \infty$, (1.3) holds with $\bar{x}_\varepsilon \in \partial\Omega_\varepsilon$ when Ω is the unit ball. To treat the general domain Ω, we define $v_\varepsilon(x) = u_\varepsilon(\varepsilon x)$, $x \in R^n \backslash \Omega$. Then v_ε is a ground state of

$$\begin{cases} \dfrac{1}{\varepsilon^2}\Delta v - v + v^p = 0, & x \in R^n \backslash \bar{\Omega} \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

By slightly modifying the arguments of Ni and Takagi [8] (who considered (1.7) in the interior of Ω), we can show that for large ε, v_ε has one and only one local maximum point which has to be on $\partial\Omega$. Thus for such ε, u_ε has one and only one local maximum point $\bar{x}_\varepsilon$ which has to be on $\partial\Omega_\varepsilon$. From this we can also show that (1.3) still holds with $\bar{x}_\varepsilon \in \partial\Omega_\varepsilon$ for general Ω.

In connection to (1.7), we mention that Z.-Q. Wang [15] obtained a multiplicity result for large ε. We also mention that (1.7) (or equivalently (1.1)) with p equal to the critical Sobolev exponent $(n+2)/(n-2)$ has been studied by Pan and Wang in [12]. In contrast to Esteban's existence result for subcritical p, in the critical case, (1.7) (or (1.1)) may or may not have a ground state. For example, if Ω is close to a ball in the C^2 sense, (1.7) and (1.1) have no ground states for every $\varepsilon > 0$ (the same is true if the mean curvature of $\partial\Omega$ is nonnegative and if $n > 3$ and ε is large), while if the mean curvature of $\partial\Omega$ is negative somewhere, (1.7) and (1.1) have a ground state for every $\varepsilon > 0$. See [12] for more on the existence and behavior of ground states of (1.7) with critical exponent p.

Throughout this paper, unless otherwise stated, the letter C will always denote various generic constants which are independent of ε, for ε sufficiently small.

2. NOTATION AND PRELIMINARIES

The first result in this section asserts that the ground state u_ε of (1.1) is bounded for every $\varepsilon > 0$. To prove this, we use the Moser iteration on (1.1) as in Theorem 3, Step 2 of [7] (where (1.7) in the interior domain case was handled, but now we take Ω in that paper to be our Ω_ε and take $d = 1$). Then we have

$$\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leqslant C\, \|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}. \tag{2.1}$$

Here since the boundary of Ω_ε is changing, we have to be careful about the Sobolev inequality used to prove (2.1): By Proposition 2.1 in [12],

$$0 < S(\Omega_1) = \inf_{\varphi \in H^1(\Omega_1)} \frac{\|\nabla\varphi\|^2_{L^2(\Omega_1)}}{\|\varphi\|^2_{L^{2n/(n-2)}(\Omega_1)}} \tag{2.2}$$

(note $\Omega_1 = \Omega^c$). It is easy to check that $S(\Omega_\varepsilon)$, defined in the obvious way, is equal to $S(\Omega_1)$. This Sobolev inequality makes the arguments in [7] valid in our case. Now to obtain a bound for $\|u_\varepsilon\|_{L^p(\Omega_\varepsilon)}$, $\varepsilon > 0$, we merely need to use the following result of [1], which says

$$0 < I_\varepsilon(u_\varepsilon) < I, \tag{2.3}$$

where I is the energy of w, i.e.,

$$I = \frac{\int_{R^n} (|\nabla w|^2 + w^2)}{(\int_{R^n} |w|^{p+1})^{2/(p+1)}}. \tag{2.4}$$

From (2.3) we have

$$I > I_\varepsilon(u_\varepsilon) = \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)}^{2(p-1)/(p+1)} = \|u_\varepsilon\|_{L^{p+1}(\Omega_\varepsilon)}^{p-1}.$$

We have thus proved

LEMMA 2.1. *There exists constant C such that*

$$\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \leqslant C \qquad \text{for all} \quad \varepsilon > 0. \tag{2.5}$$

Remark. The arguments in [7] actually imply that Lemma 2.1 holds for any positive solution of (1.1) and for ε bounded below from zero. Lemma 2.1 in the special case when Ω is a ball is proved in [2] where the symmetry was used.

The next result asserts that the ground state u_ε of (1.1), when translated to a local maximum point, converges to the unique solution w of (1.4). Let A_ε be the set of all local maximum points of u_ε. For each fixed $\varepsilon > 0$, A_ε is bounded because $u_\varepsilon(\infty) = 0$ and because any local maximum value of u_ε is bigger than 1. As $\varepsilon \to 0$, A_ε could remain either bounded or become unbounded.

LEMMA 2.2. *For any $\varepsilon > 0$, let x_ε be a point in A_ε which has the largest distance to the origin among all points in A_ε. As $\varepsilon \to 0$, we have*

$$\|u_\varepsilon(\cdot) - w(\cdot - x_\varepsilon)\|_{H^1(\Omega_\varepsilon)} \to 0. \tag{2.6}$$

Moreover $I_\varepsilon(u_\varepsilon) \to I$, where I is given by (2.4).

Remark. As mentioned in Section 1, (1.3) is proved in [1] by using the concentration-compactness principle developed in [6]. In [1], Ω is taken to be a ball, which is not essential but there is a delicate point concerning the Sobolev inequality on Ω_ε when Ω is arbitrary. (See (2.2) and the proof below.) Nonetheless, the $\bar{x}_\varepsilon$ in (1.3) yielded by the concentration-compactness method may not be taken to be x_ε in (2.5) without further arguments. Our proof below does not rely on the concentration-compactness principle.

Proof of Lemma 2.2. We first show that $I_\varepsilon(u_\varepsilon) \to I$ as $\varepsilon \to 0$. There are two cases to consider.

Case 1. As $\varepsilon \to 0$, $|x_\varepsilon| \to \infty$. Put $Z_\varepsilon(x) = u_\varepsilon(x + x_\varepsilon)$, $x \in \Omega_\varepsilon - x_\varepsilon$. Then Z_ε satisfies

$$\begin{cases} \Delta Z_\varepsilon - Z_\varepsilon + Z_\varepsilon^{\,p} = 0, & x \in \Omega_\varepsilon - x_\varepsilon \\ \dfrac{\partial Z_\varepsilon}{\partial \nu} = 0 & \text{on } \partial(\Omega_\varepsilon - x_\varepsilon) \\ Z_\varepsilon(0) > 1, & Z_\varepsilon \leqslant C \end{cases} \tag{2.7}$$

and

$$I_\varepsilon(u_\varepsilon) = \|Z_\varepsilon\|^{2(p-1)/(p+1)}_{H^1(\Omega_\varepsilon - x_\varepsilon)} = \|Z_\varepsilon\|^{p-1}_{L^{p+1}(\Omega_\varepsilon - x_\varepsilon)}. \tag{2.8}$$

Applying the elliptic interior estimates, we have, after passing to a subsequence, $Z_\varepsilon \to \bar{w}$ in $C^2_{\mathrm{loc}}(R^n)$, where $\bar{w}$ is a positive solution of the differential equation in (1.4) with $\bar{w}(0) > 1$, $\nabla\bar{w}(0) = 0$ (since $\nabla Z_\varepsilon(0) = 0$). Moreover by (2.3), (2.7) and (2.8), $\bar{w} \in H^1(R^n)$ and hence $\bar{w}(\infty) = 0$. Thus $\bar{w}$ is just the unique solution w of (1.4). Now by (2.8) and Fatou's lemma, we have

$$\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) \geqslant \|w\|^{p-1}_{L^{p+1}(R^n)} \ (= I)$$

Combining this with (2.3), we have $I_\varepsilon(u_\varepsilon) \to I$ as $\varepsilon \to 0$.

Case 2. x_ε and hence A_ε remains bounded as $\varepsilon \to 0$. Then u_ε decays at ∞ uniformly for all small ε. Otherwise, there exists a sequence of points q_i and a constant $\delta > 0$ such that $u_{\varepsilon_i}(q_i) \geqslant \delta$ and $|q_i| \to \infty$ as $\varepsilon_i \to 0$. Then as in Case 1, $u_{\varepsilon_i}(x + q_i) \to \bar{w}$ in $C^2_{\mathrm{loc}}(R^n)$ as $\varepsilon_i \to 0$. But this $\bar{w}$ may not be w, rather $\bar{w} = w(x - Q)$ for some point $Q \in R^n$. Since $\bar{w}$ has maximum at $x = Q$, $u_{\varepsilon_i}(x)$ should have at least a local maximum point near $Q + q_i$, which diverges to ∞ as $\varepsilon_i \to 0$, contradicting the assumption that A_ε is bounded as $\varepsilon \to 0$.

Now by the comparison principle, it is easy to show that there exist positive constants C and a such that

$$0 < u_\varepsilon(x) \leqslant Ce^{-a|x|} \qquad \text{for} \quad x \in \Omega_\varepsilon \quad \text{and all small } \varepsilon. \tag{2.9}$$

For any sequence $\varepsilon_i \to 0$, there exists a subsequence, still denoted by ε_i, such that $x_{\varepsilon_i} \to$ some point $-x_0$ because x_ε is bounded in Case 2. Again as in Case 1, by the interior estimates, we have $Z_{\varepsilon_i} \to \bar{w} \geqslant 0$ in $C^2_{\mathrm{loc}}(R^n \setminus \{x_0\})$, where $\bar{w}$ satisfies the differential equation in (1.4) for $x \neq x_0$. By (2.3) and (2.8) again, $\bar{w} \in H^1(R^n)$. Then it is standard to show that $\bar{w}$ is a weak solution and hence classical solution of (1.4) in the whole R^n, especially at $x = x_0$. We claim that $\bar{w} \not\equiv 0$ and hence by the strong maximum principle, $\bar{w} > 0$ in R^n (note that $Z_\varepsilon(0) = u_\varepsilon(x_\varepsilon) > 1$ does not imply that $\bar{w}(0) > 1$ because x_0 might be equal to 0). To see this, assume that $\bar{w} \equiv 0$. Then by (2.8) and (2.9) and Lebesgue's Dominated Convergence Theorem, we have

$$I_{\varepsilon_i}(u_{\varepsilon_i}) \to 0. \tag{2.10}$$

But $I_\varepsilon(u_\varepsilon)$ is bounded below by a positive constant which is independent of $\varepsilon > 0$:

$$\begin{aligned} I_\varepsilon(u_\varepsilon) &= \frac{\int_{\Omega_\varepsilon}(|\nabla u_\varepsilon|^2 + u_\varepsilon{}^2)\,dx}{(\int_{\Omega_\varepsilon} u_\varepsilon^{p+1}\,dx)^{2p+1}} \\ &\geqslant \frac{\int_{\Omega_\varepsilon}(|\nabla u_\varepsilon|^2 + u_\varepsilon{}^2)\,dx}{\|u_\varepsilon\|^{2\theta}_{L^{2n/(n-2)}(\Omega_\varepsilon)}\,\|u_\varepsilon\|^{2(1-\theta)}_{L^2(\Omega_\varepsilon)}} \\ &\geqslant \frac{(\int_{\Omega_\varepsilon}|\nabla u_\varepsilon|^2\,dx)^\theta}{\|u_\varepsilon\|^{2\theta}_{L^{2n/(n-2)}(\Omega_\varepsilon)}} \\ &\geqslant S^\theta(\Omega_\varepsilon) \\ &= S^\theta(\Omega_1) > 0 \end{aligned} \tag{2.11}$$

(see (2.2)), where θ introduced in the second inequality is defined by

$$2(1-\theta) + \theta 2n/(n-2) = p+1.$$

We have thus shown $\bar{w}$ satisfies everything in (1.4), except $\nabla\bar{w}(0) = 0$. We claim that $\bar{w} = w$. Indeed, if $x_0 \neq 0$, then $\nabla\bar{w}(0) = 0$ because $\nabla Z_\varepsilon(0) = 0$ and $Z_{\varepsilon_i} \to \bar{w}$ in $C^2_{\mathrm{loc}}(R^n \backslash \{x_0\})$. If $x_0 = 0$, i.e., $x_{\varepsilon_i} \to 0$, then by the definition of x_ε, all local maximum points of u_{ε_i} converge to the origin as $\varepsilon_i \to 0$, so that outside any fixed neighborhood of the origin there exists no local maximum point of u_{ε_i} for ε_i small. If $\bar{w} \not\equiv w$, then by the uniqueness result [5], $\bar{w}$ is just a shift of w and hence the unique local maximum point of $\bar{w}$ is not equal to 0. Then near this point, Z_{ε_i} must have at least a local maximum point for ε_i small. This is a contradiction.

We have shown that $Z_{\varepsilon_i} \to w$ in $C^2_{\mathrm{loc}}(R^n \backslash \{x_0\})$. As in Case 1, this implies $\lim_{\varepsilon \to} I_\varepsilon(u_\varepsilon) = I$.

Now to finish the proof of Lemma 2.2, we only need to show that $\|Z_\varepsilon - w\|_{H^1(\Omega_\varepsilon - x_\varepsilon)} \to 0$ as $\varepsilon \to 0$. To this end, observe that

$$\|Z_\varepsilon - w\|^2_{H^1(\Omega_\varepsilon - x_\varepsilon)} = \|Z_\varepsilon\|^2_{H^1(\Omega_\varepsilon - x_\varepsilon)} + \|w\|^2_{H^1(\Omega_\varepsilon - x_\varepsilon)} - 2\langle Z_\varepsilon, w\rangle_{H^1(\Omega_\varepsilon - x_\varepsilon)} \tag{2.12}$$

By (2.8) and the fact that $\lim_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) = I$, we have as $\varepsilon \to 0$,

$$\|Z_\varepsilon\|^2_{H^1(\Omega_\varepsilon - x_\varepsilon)} + \|w\|^2_{H^1(\Omega_\varepsilon - x_\varepsilon)} \to 2\|w\|^2_{H^1(R^n)}. \tag{2.13}$$

Mutiplying the differential equation in (2.7) by w and integrating by parts we have as $\varepsilon \to 0$,

$$\langle Z_\varepsilon, w\rangle_{H^1(\Omega_\varepsilon - x_\varepsilon)} = \int_{\Omega_\varepsilon - x_\varepsilon} Z_\varepsilon^p w \to \int_{R^n} w^{p+1}$$

by the Lebesgue's Dominated Convergence Theorem, Lemma 2.1 and the fact that w decays exponentially at infinity. Combining this with (2.12) and (2.13), we have $\|Z_\varepsilon - w\|_{H^1(\Omega_\varepsilon - x_\varepsilon)} \to 0$ as $\varepsilon \to 0$. ∎

By Lemma 2.2, we see that $u_\varepsilon(x)$ can be approximated by $w(x-x_\varepsilon)$. But for our purpose, this approximation is not good enough since $w(x-x_\varepsilon)$ does not satisfy the Neumann boundary condition. The standard resolution to this problem is to "project" all translations of w "onto Ω_ε" and approximate u_ε by a constant multiple of such a projection of w. This idea has been used in many papers ([13], [15], [16], etc.). Let us first define the projection. Let $w_{\varepsilon, y}$ be the unique solution of

$$\begin{cases} \Delta u - u + w^p(x-y) = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon, \\ u(x) \to 0 & \text{as } \quad |x| \to \infty, \end{cases} \tag{2.14}$$

where $y \in R^n$ is regarded as a parameter, Then $w_{\varepsilon, y}$ is positive in Ω_ε and belongs to $H^1(\Omega_\varepsilon)$.

Define a manifold in $H^1(\Omega_\varepsilon)$ as follows:

$$M_\varepsilon = \{ c w_{\varepsilon, y} \mid 0 < \varepsilon \leqslant 1, \, c \in R, \, y \in R^n \}.$$

Then we have

LEMMA 2.3. $\operatorname{dist}_{H^1(\Omega_\varepsilon)}(u_\varepsilon, M_\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. In view of Lemma 2.2, we just need to show that $\|\bar{\psi}_\varepsilon\|_{H^1(\Omega_\varepsilon)} \to 0$ as $\varepsilon \to 0$, where $\bar{\psi}_\varepsilon = w_{\varepsilon, x_\varepsilon}(\cdot) - w(\cdot - x_\varepsilon)$. Note that $\bar{\psi}_\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies

$$\begin{cases} \Delta \bar{\psi}_\varepsilon - \bar{\psi}_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial \bar{\psi}_\varepsilon}{\partial \nu} = -\dfrac{\partial w(x-x_\varepsilon)}{\partial \nu} & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.15}$$

Multiplying both sides of (2.15) by $\bar{\psi}_\varepsilon$ and integrating by parts, we have

$$\int_{\Omega_\varepsilon} |\nabla \bar{\psi}_\varepsilon|^2 + \bar{\psi}_\varepsilon^2 = -\int_{\partial\Omega_\varepsilon} \bar{\psi}_\varepsilon(x) \, \frac{\partial w(x-x_\varepsilon)}{\partial \nu}. \tag{2.16}$$

We claim that there exists a constant $C > 0$ (independent of x_ε) such that for all small $\varepsilon > 0$, we have

$$|\bar{\psi}_\varepsilon(x)| \leqslant C \varepsilon^{n-1} \Gamma(x), \qquad x \in \Omega_\varepsilon, \tag{2.17}$$

where Γ is the fundamental solution of the operator $-\Delta u+u$ in R^n, with pole at 0. (Recall $\Gamma(x)$ behaves like $|x|^{2-n}$ at the origin.)

Combining (2.16) and (2.17), we have

$$\|\bar{\psi}_\varepsilon\|_{H^1(\Omega_\varepsilon)}=O(\varepsilon^n).$$

Now we prove (2.17). Let $\tilde{\psi}_\varepsilon(x)=\bar{\psi}_\varepsilon(\varepsilon x)/\varepsilon$, $x\in\Omega_1$. Then it satisfies

$$\begin{cases}\Delta\tilde{\psi}_\varepsilon-\varepsilon^2\tilde{\psi}_\varepsilon=0 & \text{in } \Omega_1,\\ \dfrac{\partial\tilde{\psi}_\varepsilon}{\partial\nu}=-\dfrac{\partial w(\varepsilon x-x_\varepsilon)}{\partial\nu} & \text{on } \partial\Omega_1.\end{cases} \tag{2.18}$$

Take a positive constant K such that $K\geqslant|\partial w(\varepsilon x-x_\varepsilon)/\partial\nu|$ for all $\varepsilon>0$ and $x\in\partial\Omega_1$. Consider the inhomogeneous Neumann problem

$$\begin{cases}\Delta\Psi=0 & \text{in } \Omega_1,\\ \dfrac{\partial\Psi}{\partial\nu}=K & \text{on } \partial\Omega_1, \quad \Psi(\infty)=0.\end{cases} \tag{2.19}$$

By the classical potential theory (see [4]), (2.20) has a unique positive solution and is given by the single layer potential

$$\Psi(x)=\int_{\partial\Omega_1}\sigma(y)\,\Gamma_0(x-y)\,dS_y$$

where σ is a continuous function on $\partial\Omega_1$, and Γ_0 is the fundamental solution of $-\Delta$ in R^n. Therefore for some constant C,

$$0<\Psi(x)\leqslant C\Gamma_0(x), \qquad x\in\Omega_1.$$

Observe that Ψ is an upper solution and $-\Psi$ is a lower solution of (2.18), and hence the comparison principle implies

$$|\tilde{\psi}_\varepsilon(x)|\leqslant\Psi(x)\leqslant C\Gamma_0(x), \qquad x\in\Omega_1, \quad \varepsilon>0.$$

Thus

$$\bar{\psi}_\varepsilon(x)\leqslant C\varepsilon\Gamma_0\left(\frac{x}{\varepsilon}\right)=C\varepsilon^{n-1}\Gamma_0(x), \qquad x\in\Omega_\varepsilon, \quad \varepsilon>0. \tag{2.20}$$

In particular, $\bar{\psi}_\varepsilon(x)/\varepsilon^{n-1}$ decays at infinity uniformly with respect to small $\varepsilon>0$. Then by the comparison principle again,

$$\bar{\psi}_\varepsilon(x)/\varepsilon^{n-1}\leqslant C\Gamma(x), \qquad |x|\geqslant 1, \quad \varepsilon>0 \ \text{ small.} \tag{2.21}$$

Combining (2.20), (2.21) and the fact that both Γ_0 and Γ have the same order of singularity $|x|^{2-n}$ at origin, we have (2.17).

Lemma 2.3 is proved. ∎

Lemma 2.3 can be used to prove the following in the standard way (see [13], [15] for similar results).

LEMMA 2.4. *For small* $\varepsilon>0$, $\operatorname{dist}_{H^1(\Omega_\varepsilon)}(u_\varepsilon, M_\varepsilon)$ *is achieved by some* $C_\varepsilon w_{\varepsilon, y_\varepsilon} \in M_\varepsilon$. *Moreover,* $C_\varepsilon \to 1$ *and* $|x_\varepsilon - y_\varepsilon| \to 0$ *as* $\varepsilon \to 0$.

Now we write for small $\varepsilon>0$,

$$u_\varepsilon = C_\varepsilon w_{\varepsilon, y_\varepsilon} + \phi_\varepsilon. \tag{2.22}$$

Then we have

$$\langle \phi_\varepsilon, w_{\varepsilon, y_\varepsilon} \rangle_{H^1(\Omega_\varepsilon)} = \left\langle \phi_\varepsilon, \left.\frac{\partial w_{\varepsilon, y}}{\partial y_i}\right|_{y=y_\varepsilon} \right\rangle_{H^1(\Omega_\varepsilon)} = 0, \tag{2.23}$$

where $i=1, ..., n$, $y=(y_1, ..., y_n)$.

LEMMA 2.5. *There exists positive constant* $\sigma>0$ *such that*

$$(p+\sigma)\int_{\Omega_\varepsilon} w_{\varepsilon, y_\varepsilon}^{p-1} \phi_\varepsilon{}^2 \leqslant \|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \tag{2.24}$$

for ε *small.*

Proof. Let $\psi_\varepsilon(x) = \phi_\varepsilon(x+y_\varepsilon)/\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon - y_\varepsilon)}$, $x \in \Omega_\varepsilon - y_\varepsilon$. Then $\|\psi_\varepsilon\|_{H^1(\Omega_\varepsilon)} = 1$. We shall argue by contradiction. If (2.24) is untrue, then there exists a sequence $\varepsilon \to 0$ (still denoted by ε) such that

$$1 \leqslant (p+o(1)) \int_{\Omega_\varepsilon - y_\varepsilon} w_{\varepsilon, y_\varepsilon}^{p-1}(x+y_\varepsilon)\, \psi_\varepsilon^2(x)\, dx \tag{2.25}$$

We assume, without loss of generality, that $y_\varepsilon \to y_0$, where y_0 is either a point in R^n or ∞. Since $\|\psi_\varepsilon\|_{H^1(\Omega_\varepsilon - y_\varepsilon)} = 1$, we have after passing to a subsequence,

$$\begin{aligned} &\psi_\varepsilon \to \text{ some } \psi_0 \in H^1(R^n), \text{ weakly in } H^1_{\mathrm{loc}}(R^n \backslash \{-y_0\}) \text{ and} \\ &\text{strongly in } L^2_{\mathrm{loc}}(R^n \backslash \{-y_0\}), \end{aligned} \tag{2.26}$$

where ψ_0 satisfies

$$\|\psi_0\|_{H^1(R^n)} \leqslant 1. \tag{2.27}$$

We claim that as $\varepsilon \to 0$,

$$\int_{\Omega_\varepsilon - y_\varepsilon} w_{\varepsilon, y_\varepsilon}^{p-1}(x + y_\varepsilon)\, \psi_\varepsilon^2(x)\, dx \to \int_{R^n} w^{p-1}(x)(\psi_0)^2\,(x)\, dx \tag{2.28}$$

Since $\|w_{\varepsilon, y_\varepsilon}(\cdot + y_\varepsilon) - w(\cdot)\|_{L^\infty(\Omega_\varepsilon - y_\varepsilon)} = \|w_{\varepsilon, y_\varepsilon}(\cdot) - w(\cdot - y_\varepsilon)\|_{L^\infty(\Omega_\varepsilon)} \to 0$ (see the proof of Lemma 2.3), to show (2.28), we just need to show that

$$\int_{\Omega_\varepsilon - y_\varepsilon} w^{p-1}(x)(\psi_\varepsilon)^2\,(x)\, dx \to \int_{R^n} w^{p-1}(x)(\psi_0)^2\,(x)\, dx$$

which follows easily from (2.26) and the fact that w decays exponentially at infinity.

Now sending $\varepsilon \to 0$ in (2.25) and using (2.28) and (2.27), we obtain

$$\|\psi_0\|^2_{H^1(R^n)} \leqslant 1 \leqslant p \int_{R^n} w^{p-1}(x)(\psi_0)^2\,(x)\, dx. \tag{2.29}$$

We claim that the following holds

$$\langle \psi_0, w\rangle_{H^1(R^n)} = \left\langle \psi_0, \frac{\partial w}{\partial x_i}\right\rangle_{H^1(R^n)} = 0, \qquad i = 1, 2, ..., n. \tag{2.30}$$

Since $w_{\varepsilon, y_\varepsilon}(\cdot + y_\varepsilon) - w(\cdot) \to 0$ in $H^1(\Omega_\varepsilon - y_\varepsilon)$, by (2.23) and (2.26), we have

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} \langle \psi_\varepsilon, w_{\varepsilon, y_\varepsilon}(\cdot + y_\varepsilon)\rangle_{H^1(\Omega_\varepsilon - y_\varepsilon)}, \\ &= \lim_{\varepsilon \to 0} \langle \psi_\varepsilon, w\rangle_{H^1(\Omega_\varepsilon - y_\varepsilon)}, \\ &= \langle \psi_0, w\rangle_{H^1(R^n)}. \end{aligned}$$

To prove the second equality in (2.30), we differentiate (2.14) with respect to y_i, $1 \leqslant i \leqslant n$, multiply the resulting equation by $\psi_\varepsilon(x - y_\varepsilon)$ and then integrate by parts, we have by (2.23),

$$\begin{aligned} 0 &= \int_{\Omega_\varepsilon} p w^{p-1}(x - y_\varepsilon)\, \psi_\varepsilon(x - y_\varepsilon)\, \frac{\partial w}{\partial x_i}(x - y_\varepsilon)\, dx \\ &= \int_{\Omega_\varepsilon - y_\varepsilon} p w^{p-1}(x)\, \psi_\varepsilon(x)\, \frac{\partial w}{\partial x_i}(x)\, dx. \end{aligned}$$

Sending $\varepsilon \to 0$ and using (2.26) and the fact that w and $|\nabla w|$ decay exponentially fast at infinity, we have

$$0 = \int_{R^n} p w^{p-1}(x)\, \psi_0(x) \frac{\partial w}{\partial x_i}(x)\, dx. \tag{2.31}$$

On the other hand, differentiating (1.4) with respect to x_i, we have

$$\Delta\left(\frac{\partial w}{\partial x_i}\right) - \frac{\partial w}{\partial x_i} + p w^{p-1} \frac{\partial w}{\partial x_i} = 0. \tag{2.32}$$

Multiplying both sides of (2.32) by ψ_0 and then integrating by parts, we are led to, in view of (2.31), the second equality in (2.30).

Now consider the following eigenvalue problem in $H^1(R^n)$

$$-\Delta v + v = \mu w^{p-1} v \qquad \text{in } R^n, \quad v \in H^1(R^n) \tag{2.33}$$

for which we have (see Lemma 4.1 of [15])

LEMMA 2.6. *The eigenvalues of (2.33) form a discrete set $\mu_1 < \mu_2 < \mu_3 < \cdots \to \infty$ and are given by*

$$\mu_i = \inf_{v \perp_{H^1} V_k,\, 0 \leqslant k \leqslant i-1} \frac{\int_{R^n} (|\nabla v|^2 + v^2)}{\int_{R^n} w^{p-1} v^2}, \qquad i = 1, 2, \ldots \tag{2.34}$$

where V_{i-1} is the eigenspace corresponding to μ_{i-1} with the understanding $V_0 = \{0\}$. Moreover, $\mu_1 = 1$, $V_1 = \operatorname{span}\{w\}$, $\mu_2 = p$, $V_2 = \operatorname{span}\{\partial w/\partial x_1, \ldots, \partial w/\partial x_n\}$.

Now by (2.30), ψ_0 is orthogonal to V_1 and V_2 and hence in virtue of the above lemma,

$$p = \mu_2 < \mu_3 \leqslant \frac{\int_{R^n} (|\nabla \psi_0|^2 + (\psi_0)^2)}{\int_{R^n} w^{p-1} (\psi_0)^2},$$

contradicting (2.29). The proof of Lemma 2.5 is complete. ∎

3. THE SHIFT OF GROUND STATES

Fix a point $P \in R^N$. Let $c_\Omega(P) = |P|^2 + 1/|\Omega| \int_{\Omega^c} |\nabla_y (P \cdot U(y))|^2\, dy$, where $U = (U_1, \ldots, U_n)$ and U_i is the unique solution of the following problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^c, \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty, \\ \dfrac{\partial u}{\partial \nu} = \nu_i & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where ν_i is the ith component of ν (which is the normal of $\partial\Omega$, pointing into Ω). When Ω is the unit ball, it is easy to see that

$$U_i(x) = -\frac{1}{n-1}\,|x|^{-n}\,x_i, \qquad i=1, ..., n.$$

We compute

$$\int_{\Omega^c} |\nabla U_i|^2 = \int_{\partial\Omega} U_i \frac{\partial U_i}{\partial \nu}$$

$$= -\frac{1}{n-1}\int_{\partial\Omega} \nu_i x_i = \frac{1}{n-1}\,|\Omega|;$$

$$\int_{\Omega^c} \nabla U_i \cdot \nabla U_j = 0, \qquad i \neq j.$$

So when Ω is the unit ball, we have

$$c_\Omega(P) = \frac{n}{n-1}\,|P|^2.$$

Set

$$F_\Omega(P) = c_\Omega(P)\,\frac{|w'|^2\,(|P|)}{|P|^2} + w^2(|P|) - \frac{2}{p+1}\,w^{p+1}(|P|). \tag{3.2}$$

We remark that F is continuous and a global maximum point x_0 of $F_\Omega(P)$ exists with $0 < |x_0| < \infty$. To see this, observe that since $E(r) := (w')^2 - w^2 + 2/(p+1)\,w^{p+1}$ is decreasing and decays at $r=\infty$, $E(0)>0$ and hence $F_\Omega(0) = w^2(0) - 2/(p+1)\,w^{p+1}(0) < 0$. Clearly F_Ω takes positive values and $F_\Omega(\infty)=0$. Thus F_Ω achieves its maximum at a finite point x_0. We do not know if this point is unique though.

In this section, we shall prove the following.

THEOREM 3.1. *For each sequence $\varepsilon_k \to 0$, there exist a subsequence (still denoted by ε_k) and a nonzero point $x_0 \in R^n$ so that x_0 is a global maximum point of $F_\Omega(x)$, and*

$$u_{\varepsilon_k}(\cdot) - w(\cdot - x_0) \to 0 \qquad \text{in } H^1(\Omega_{\varepsilon_k}); \qquad C^2_{\mathrm{loc}}(R^n\backslash\{0\}), \tag{3.3}$$

Moreover, all local maximum points of u_ε other than x_ε (if any) converge to the origin as $\varepsilon \to 0$. (For a definition of x_ε, see Lemma 2.2.).

Proof. We divide the proof into three steps.

Step 1. We shall show that for small ε,

$$I_\varepsilon(u_\varepsilon) \geqslant I_\varepsilon(w_\varepsilon) + O(1) \int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon + O(\|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}), \tag{3.4}$$

where $w_\varepsilon = w_{\varepsilon, y_\varepsilon}$ is defined by Lemma 2.4 and ϕ_ε is defined in (2.22).

By the elementary inequality,

$$|(1+\xi)^{p+1} - 1 - (p+1)\xi| = O(\xi^2 + |\xi|^{p+1}), \qquad \xi \geqslant -1, \tag{3.5}$$

we get

$$\begin{aligned}
\int_{\Omega_\varepsilon} u_\varepsilon^{p+1} &= \int_{\Omega_\varepsilon} (C_\varepsilon w_\varepsilon + \phi_\varepsilon)^{p+1}\, dx \\
&= \int_{\Omega_\varepsilon} [C_\varepsilon^{p+1} w_\varepsilon^{p+1} + (p+1)\, C_\varepsilon^p w_\varepsilon^p \phi_\varepsilon] + O(1) \int_{\Omega_\varepsilon} (w_\varepsilon^{p-1} \phi_\varepsilon^2 + \phi_\varepsilon^{p+1}) \\
&= \int_{\Omega_\varepsilon} [C_\varepsilon^{p+1} w_\varepsilon^{p+1} + (p+1)\, C_\varepsilon^p w_\varepsilon^p \phi_\varepsilon] + O(\|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}),
\end{aligned} \tag{3.6}$$

where in the last equality, we also used the fact that $\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)} = o(1)$ (Lemmas 2.3 and 2.4) and the Sobolev inequality

$$S^\theta(\Omega_1) < \inf_{u \in H^1(\Omega_\varepsilon)} I_\varepsilon(u), \qquad \varepsilon > 0$$

(see (2.11)). Now by (2.23), (3.6), Lemma 2.4 and Taylor's Theorem

$$\begin{aligned}
I_\varepsilon(u_\varepsilon) &= \frac{\|C_\varepsilon w_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} + \|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}}{(\int_{\Omega_\varepsilon} C_\varepsilon^{p+1} w_\varepsilon^{p+1} + (p+1)\, C_\varepsilon^p w_\varepsilon^p \phi_\varepsilon + O(\|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}))^{2/(p+1)}} \\
&\geqslant C_\varepsilon^2 \|w_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \left[C_\varepsilon^{-2} \left(\int_{\Omega_\varepsilon} w_\varepsilon^{p+1} \right)^{-2/(p+1)} + O(1) \int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon \right. \\
&\quad \left. + O(\|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}) \right] \\
&= I_\varepsilon(w_\varepsilon) + O(1) \int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon + O(\|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}).
\end{aligned}$$

Step 2. We shall prove that

$$\int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon = o(\varepsilon^n) = \|\phi_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}. \tag{3.7}$$

Multiplying (1.1) (with $u = u_\varepsilon = C_\varepsilon w_\varepsilon + \phi_\varepsilon$) by ϕ_ε, integrating by parts and using the orthogonality $\phi_\varepsilon \perp_{H^1(\Omega_\varepsilon)} w_\varepsilon$, we have

$$\int_{\Omega_\varepsilon} (|\nabla \phi_\varepsilon|^2 + \phi_\varepsilon^2) = \int_{\Omega_\varepsilon} u_\varepsilon^p \phi_\varepsilon. \tag{3.8}$$

On the other hand, by using the elementary inequality

$$|(1+\xi)^p - 1 - p\xi| = \begin{cases} O(|\xi|^p), & \xi \geqslant -1, \quad \text{if} \quad 1 < p < 2, \\ O(|\xi|^2 + |\xi|^p), & \xi \geqslant -1, \quad \text{if} \quad p \geqslant 2, \end{cases}$$

we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} u_\varepsilon^p \phi_\varepsilon &= \int_{\Omega_\varepsilon} (C_\varepsilon w_\varepsilon + \phi_\varepsilon)^p \, \phi_\varepsilon, \\ &= \int_{\Omega_\varepsilon} [C_\varepsilon^p w_\varepsilon^p \phi_\varepsilon + p C_\varepsilon^{p-1} w_\varepsilon^{p-1} \phi_\varepsilon^2 + O(1)(w_\varepsilon^{p-\sigma} \, |\phi_\varepsilon|^3 + |\phi_\varepsilon|^{p+1})], \\ &= C_\varepsilon^p \int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon + p C_\varepsilon^{p-1} \int_{\Omega_\varepsilon} w_\varepsilon^{p-1} \phi_\varepsilon^2 + O(\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)}^{1+\sigma}), \end{aligned}$$

where $\sigma = \min(p, 2)$. This and (3.8) yield

$$\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 = C_\varepsilon^p \int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon + p C_\varepsilon^{p-1} \int_{\Omega_\varepsilon} w_\varepsilon^{p-1} \phi_\varepsilon^2 + O(\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)}^{1+\sigma}). \tag{3.9}$$

From this and Lemmas 2.4 and 2.5 it follows

$$\int_{\Omega_\varepsilon} w_\varepsilon^{p-1} \phi_\varepsilon^2 \leqslant O(1) \int_{\Omega_\varepsilon} w_\varepsilon^{p-1} \phi_\varepsilon$$

and hence

$$\|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)}^2 \leqslant O(1) \int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon. \tag{3.10}$$

We proceed to show that $\int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon$ is $o(\varepsilon^n)$, then step 2 is finished. Let $\eta_\varepsilon(x) = w_\varepsilon(x) - w(x - y_\varepsilon)$. Then it satisfies

$$\begin{cases} \Delta \eta_\varepsilon - \eta_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial \eta_\varepsilon}{\partial \nu} = -\dfrac{\partial w(\cdot - y_\varepsilon)}{\partial \nu} & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3.11}$$

and

$$|\eta_\varepsilon|(x) = O(\varepsilon^{n-1}\Gamma(x)), \qquad x \in \Omega_\varepsilon \tag{3.12}$$

(see (2.17)).

Suppose the ball centered at 0 with radius δ is contained in Ω. Then

$$\begin{aligned}
\int_{\Omega_\varepsilon} \eta_\varepsilon^2(x)\, dx &\leqslant O(\varepsilon^{2n-2}) \int_{|x| \geqslant \delta\varepsilon} \Gamma^2(x)\, dx \\
&= O(\varepsilon^{2n-2}) \left(\int_{\delta\varepsilon \leqslant |x| \leqslant 1} + \int_{|x| \geqslant 1} \right) \Gamma^2(x)\, dx \\
&= O(\varepsilon^{2n-2}) \left(\int_{\delta\varepsilon}^1 r^{3-n}\, dr \right) + O(\varepsilon^{2n-2}) \\
&= O(\varepsilon^{2n-2})\, \varepsilon^{4-n} + O(\varepsilon^{2n-2}) \\
&= o(\varepsilon^n)
\end{aligned}$$

Now using orthogonality $\phi_\varepsilon \perp_{H^1(\Omega_\varepsilon)} w_\varepsilon$ and (3.10), we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon &= \int_{\Omega_\varepsilon} \phi_\varepsilon(-\Delta w_\varepsilon + w_\varepsilon) + \int_{\Omega_\varepsilon} \phi_\varepsilon(\Delta w_\varepsilon - w_\varepsilon + w_\varepsilon^p) \\
&= \int_{\Omega_\varepsilon} \phi_\varepsilon(w_\varepsilon^p - w^p(x - y_\varepsilon)) \\
&\leqslant O(1) \int_{\Omega_\varepsilon} |\phi_\varepsilon|\, |\eta_\varepsilon| \\
&\leqslant O(1)\, \|\phi_\varepsilon\|_{H^1(\Omega_\varepsilon)}\, \|\eta_\varepsilon\|_{L^2(\Omega_\varepsilon)}.
\end{aligned}$$

This and (3.10) yield

$$\int_{\Omega_\varepsilon} w_\varepsilon^p \phi_\varepsilon \leqslant O(1)\, \|\eta_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = o(\varepsilon^n).$$

Step 2 is done.

Step 3. Fix any point $P \in R^n$, let $w_{\varepsilon, P}$ be defined by (2.14). Then we have

LEMMA 3.2. *For ε sufficiently small, we have*

$$I_\varepsilon(w_{\varepsilon, P}) = I(w) - \alpha_1 \varepsilon^n F_\Omega(P) + o(\varepsilon^n), \tag{3.13}$$

where $\alpha_1 = |\Omega| (\int_{R^n} w^{p+1})^{-2/(p+1)}$, $F_\Omega(P)$ *is defined by* (3.2), *and* $o(\varepsilon^n)$ *is uniform with respect to all* $P \in R^n$.

We delay the proof of this lemma until the end of this section but use it to finish the proof of Theorem 3.1 now.

By Step 1 and Step 2, we have

$$I_\varepsilon(u_\varepsilon) \geqslant I_\varepsilon(w_\varepsilon) + o(\varepsilon^n).$$

From this and Lemma 3.2 it follows

$$I_\varepsilon(u_\varepsilon) \geqslant I(w) - \varepsilon^n \alpha_1 F_\Omega(x_\varepsilon) + o(\varepsilon^n). \tag{3.14}$$

On the other hand, since u_ε is a ground state solution, we have for $P \in R^n$

$$I_\varepsilon(u_\varepsilon) \leqslant I_\varepsilon(w_{\varepsilon, P})$$

and hence

$$I_\varepsilon(u_\varepsilon) \leqslant I(w) - \alpha_1 \varepsilon^n \max_{P \in R^n} F_\Omega(P) + o(\varepsilon^n). \tag{3.15}$$

Comparing (3.14) and (3.15), we obtain

$$\lim_{\varepsilon \to 0} F(x_\varepsilon) \geqslant \max_{P \in R^n} F_\Omega(P).$$

Hence if $x_\varepsilon \to x_0$, then x_0 is a global maximum point of $F_\Omega(x)$. As noticed at the beginning of this section, we have $0 < |x_0| < \infty$. Combining these with Lemma 2.2 and the elliptic interior estimates, we obtain (3.3).

Since $u_\varepsilon \to w(\cdot - x_0)$ in $C^2_{\text{loc}}(R^n \backslash \{0\})$ and $w(\cdot - x_0)$ has a unique non-degenerate maximum at x_0, all local maximum points of u_ε other than x_ε converge to 0 as $\varepsilon \to 0$. Theorem 3.1 is thus proved. ∎

Finally we give

Proof of Lemma 3.2. Define $\varphi_{\varepsilon, P}(x) = w(x - P) - w_{\varepsilon, P}(x)$. Then it satisfies

$$\begin{cases} \Delta u - u = 0 & \text{in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial w(\cdot - P)}{\partial \nu} & \text{on } \partial \Omega_\varepsilon. \end{cases} \tag{3.16}$$

Then by the definition of $w_{\varepsilon, P}$, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} |\nabla w_{\varepsilon, P}(x)|^2 + \int_{\Omega_\varepsilon} |w_{\varepsilon, P}(x)|^2
&= \int_{\Omega_\varepsilon} w^p(x-P)\, w_{\varepsilon, P}(x) \\
&= \int_{\Omega_\varepsilon} w^p(x-P)[w(x-P) - \varphi_{\varepsilon, P}(x)] \\
&= \int_{\Omega_\varepsilon} w^{p+1}(x-P) - \int_{\Omega_\varepsilon} w^p(x-P)\, \varphi_{\varepsilon, P}(x) \\
&= I_1 - I_2, \qquad (3.17)
\end{aligned}$$

where I_1 and I_2 are defined at the last equality. Note that

$$I_1 = \int_{\Omega_\varepsilon} w^{p+1}(x-P)\, dx = \int_{R^n} w^{p+1}(x-P)\, dx - \varepsilon^n w^{p+1}(|P|)\, |\Omega| + o(\varepsilon^n). \qquad (3.17)$$

For I_2, we have

$$\begin{aligned}
I_2 &= \int_{\Omega_\varepsilon} (w_{\varepsilon, P}(x) - \Delta w_{\varepsilon, P}(x))\, \varphi_{\varepsilon, P}(x) + (\Delta \varphi_{\varepsilon, P}(x) - \varphi_{\varepsilon, P}(x))\, w_{\varepsilon, P}(x) \\
&= \int_{\partial\Omega_\varepsilon} \frac{\partial \varphi_{\varepsilon, P}(x)}{\partial \nu}\, w_{\varepsilon, P}(x) \\
&= \int_{\partial\Omega_\varepsilon} \frac{\partial \varphi_{\varepsilon, P}(x)}{\partial \nu}\, w(x-P) - \int_{\partial\Omega_\varepsilon} \frac{\partial \varphi_{\varepsilon, P}(x)}{\partial \nu}\, \varphi_{\varepsilon, P}(x) \\
&= I_{21} - I_{22},
\end{aligned}$$

where I_{21}, I_{22} are defined at the last equality.

I_{21} can be estimated by using the Taylor expansion and the Divergence Theorem as follows (we use the notation w_i for $\partial w/\partial x_i$, w_{ij} for $\partial^2 w/\partial x_i \partial x_j$ and the summation convention).

$$\begin{aligned}
I_{21} &= \int_{\partial\Omega_\varepsilon} \frac{\partial w(x-P)}{\partial \nu}\, w(x-P)\, dS_x \\
&= \varepsilon^{n-1} \int_{\partial\Omega} \nabla w(\varepsilon y - P) \cdot \nu w(\varepsilon y - P)\, dS_y
\end{aligned}$$

$$= \varepsilon^{n-1} \int_{\partial\Omega} (w_i(-P) + \varepsilon w_{ij}(-P)\, y_j + O(\varepsilon^2))\, v_i(w(-P)$$

$$+ \varepsilon w_j(-P) y_j + O(\varepsilon^2))\, dS_y$$

$$= \varepsilon^n \int_{\partial\Omega} (w_i(-P)\, v_i w_j(-P)\, y_j + w(-P)\, w_{ij}(-P)\, y_j v_i)\, dS_y + O(\varepsilon^{n+1})$$

$$= -\varepsilon^n \int_{\Omega} (w_i(-P)\, w_j(-P)\, \delta_{ij} + w(-P)\, w_{ij}(-P)\, \delta_{ij})\, dy + O(\varepsilon^{n+1})$$

$$= -\varepsilon^n\, |\Omega|\, [\,|\nabla w(-P)|^2 + w(-P)\, \Delta w(-P)] + o(\varepsilon^n)$$

$$= -\varepsilon^n\, |\Omega|\, [(w'(|P|))^2 + (w(|P|))^2 - (w(|P|))^{p+1}] + o(\varepsilon^n).$$

To estimate I_{22}, we define $\tilde{\varphi}_{\varepsilon, P}(y) = \varphi_{\varepsilon, P}(\varepsilon y)$, $y \in \Omega$. Then it satisfies

$$\begin{cases} \Delta \tilde{\varphi}_{\varepsilon, P}(z) - \varepsilon^2 \tilde{\varphi}_{\varepsilon, P} = 0 & \text{in } \Omega^c \\ \dfrac{\partial \tilde{\varphi}_{\varepsilon, P}}{\partial v} = \varepsilon \nabla w(-P) \cdot v + O(\varepsilon^2) & \text{on } \partial\Omega \end{cases} \tag{3.18}$$

We claim that as $\varepsilon \to 0$

$$\tilde{\varphi}_{\varepsilon, P}(y) = \varepsilon(1 + O(\varepsilon))\, \nabla w(-P) \cdot U(y) \tag{3.19}$$

uniformly for $y \in \partial\Omega$ and $P \in R^n$, where U is defined by (3.1). In fact, let $\Psi_{\varepsilon, P}(y) = (\tilde{\varphi}_{\varepsilon, P}(y) - \varepsilon \nabla w(-P) \cdot U(y))/\varepsilon^2$, then it satisfies

$$\begin{cases} \Delta \Psi_{\varepsilon, P} - \tilde{\varphi}_{\varepsilon, P} = 0 & \text{in } \Omega^c \\ \dfrac{\partial \Psi_{\varepsilon, P}}{\partial v} = O(1) & \text{on } \partial\Omega, \quad \Psi_{\varepsilon, P}(\infty) = 0. \end{cases} \tag{3.20}$$

By using the arguments leading to (2.20), we have

$$|\tilde{\varphi}_{\varepsilon, P}(y)| \leqslant C\varepsilon \Gamma_0(y), \qquad y \in \Omega^c.$$

Let Ψ be the unique positive solution of

$$\begin{cases} \Delta \Psi + \Gamma_0 = 0 & \text{in } \Omega^c \\ \dfrac{\partial \Psi}{\partial v} = M & \text{on } \partial\Omega, \quad \Psi(\infty) = 0. \end{cases}$$

where M is an upper bound for the absolute value of the $O(1)$ term in (3.20) (because of the decay of Γ_0, Ψ exists). Observe that for ε small and all $P \in R^n$, Ψ is an upper solution and $-\Psi$ a lower solution of (3.20). Thus by the comparison principle, we have $|\Psi_{\varepsilon, P}| \leqslant \Psi$ in Ω^c, from which (3.19) follows.

Now by (3.19), Taylor's expansion, (3.1) and the Divergence Theorem, we compute,

$$
\begin{aligned}
I_{22} &= \int_{\partial \Omega_\varepsilon} \frac{\partial \varphi_{\varepsilon, P}}{\partial \nu} \varphi_{\varepsilon, P} \\
&= \varepsilon^n \int_{\partial \Omega} (\nabla w(-P) \cdot U(y))(\nabla w(-P) \cdot \nu)\, dS_y + O(\varepsilon^{n+1}) \\
&= \varepsilon^n \int_{\partial \Omega} (\nabla w(-P) \cdot U(y)) \left(\nabla w(-P) \cdot \frac{\partial U(y)}{\partial \nu} \right) dS_y + O(\varepsilon^{n+1}) \\
&= \varepsilon^n \int_{\Omega^c} |\nabla_y (\nabla w(-P) \cdot U(y))|^2\, dy + O(\varepsilon^{n+1}) \\
&= \varepsilon^n \frac{(w'(|P|))^2}{|P|^2} \int_{\Omega^c} |\nabla_y (P \cdot U(y))|^2\, dy + o(\varepsilon^n)
\end{aligned}
$$

Now combining the estimates for I_{21} and I_{22}, we infer

$$
I_2 = -\varepsilon^n \left\{ w^2(|P|) - w^{p+1}(|P|) + c_\Omega(P) \frac{(w'(|P|))^2}{|P|^2} \right\} |\Omega| + o(\varepsilon^n).
$$

Observe that

$$
\begin{aligned}
\int_{\Omega_\varepsilon} w_{\varepsilon, P}^{p+1}(x) &= \int_{\Omega_\varepsilon} w^{p+1}(x-P) - (p+1) \int_{\Omega_\varepsilon} w^p(x-P)\, \varphi_{\varepsilon, P}(x) \\
&\quad + O(\|\varphi_{\varepsilon, P}\|^2_{L^2(\Omega_\varepsilon)}) \\
&= I_1 - (p+1)\, I_2 + o(\varepsilon^n)
\end{aligned}
$$

(see the estimates following (3.12)).

Finally, we are ready to compute

$$
\begin{aligned}
I_\varepsilon(w_{\varepsilon, P}) &= \frac{\int_{\Omega_\varepsilon} (|\nabla w_{\varepsilon, P}|^2 + w_{\varepsilon, P}^2)}{(\int_{\Omega_\varepsilon} w_{\varepsilon, P}^{p+1})^{2/(p+1)}} \\
&= \frac{I_1 - I_2}{(I_1 - (p+1)\, I_2 + o(\varepsilon^n))^{2/(p+1)}} \\
&= I_1^{(p-1)/(p+1)} + I_1^{-2/(p+1)} I_2 + o(\varepsilon^n)
\end{aligned}
$$

$$
\begin{aligned}
&= I(w) - \alpha_1 \varepsilon^n \left[\frac{p-1}{p+1} w^{p+1}(|P|) + w^2(|P|) - w^{p+1}(|P|) \right. \\
&\quad \left. + c_\Omega(P) \frac{(w'(|P|))^2}{|P|^2} \right] |\Omega| + o(\varepsilon^n) \\
&= I(w) - \alpha_1 \varepsilon^n F_\Omega(P) + o(\varepsilon^n).
\end{aligned}
$$

where α_1 is defined in the statement of Lemma 3.2. Lemma 3.2 is thus proved. ∎

4. THE STABILITY OF GROUND STATES

In this section, we deal with the stability question of solitary wave $e^{it}u_\varepsilon$ of the nonlinear Schrödinger Eq. (1.6). Remember now we specialize in the case when Ω is the unit ball centered at the origin. Then by Proposition 1.4 in [2], up to rotation the ground state u_ε of (1.1) has partial symmetry. A function $v(x)$ defined on Ω_ε is said to have partial symmetry if $v(x) = g(|x|, \theta)$, $\theta = \arccos(x_1/|x|)$, where g satisfies that for each fixed $r \geqslant \varepsilon$, $g(r, \theta)$ is nonincreasing for $\theta \in [0, \pi]$. The set of all such v's in $H^1(\Omega_\varepsilon)$ satisfying the Neumann boundary condition will be denoted by $H^1_*(\Omega_\varepsilon)$. We shall always choose u_ε such that $u_\varepsilon \in H^1_*(\Omega_\varepsilon)$.

The notion of $(H^1_*(\Omega_\varepsilon), G_g)$ stability of solitary wave $e^{it}u_\varepsilon$ is introduced in [2]. According to that, we say $e^{it}u_\varepsilon$ is $(H^1_*(\Omega_\varepsilon), G_g)$ stable if for any $\sigma > 0$, there exists $\delta > 0$ such that $\|u(0) - u_\varepsilon\|_{H^1(\Omega_\varepsilon)} < \delta$ implies that the solution $u(t)$ of (1.6) with initial value $u(0)$ exists globally and that

$$
\sup_{0 \leqslant t < \infty} \inf_{-\infty < s < \infty} \|u(t) - e^{i(s+t)}u_\varepsilon\|_{H^1(\Omega_\varepsilon)} < \sigma.
$$

THEOREM 4.1. *For ε small, the solitary wave $e^{it}u_\varepsilon$ of (1.6) is $(H^1_*(\Omega_\varepsilon), G_g)$ stable, provided $1 < p < 1 + 4/n$.*

Proof. As remarked on p. 193 of [2], we only need to show that the operator

$$
M_\varepsilon = -\Delta + 1 - p u_\varepsilon^{p-1}
$$

has trivial nullspace on $H^1_*(\Omega_\varepsilon)$ for ε small. Suppose otherwise, then there exist a sequence of $\varepsilon \to 0$ and $\varphi_\varepsilon \in H^1_*(\Omega_\varepsilon)$ such that $\|\varphi_\varepsilon\|_{H^1(\Omega_\varepsilon)} = 1$ and

$$
\begin{cases}
\Delta\varphi_\varepsilon - \varphi_\varepsilon + p u_\varepsilon^{p-1}\varphi_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\
\dfrac{\partial \varphi_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon.
\end{cases}
\tag{4.1}
$$

After passing to a subsequence, we have

$$\varphi_\varepsilon \to \text{some } \varphi_0, \qquad \text{weakly in } H^1_{\text{loc}}(R^n \backslash \{0\}), \quad \text{strongly in } L^2_{\text{loc}}(R^n \backslash \{0\}), \tag{4.2}$$

where $\varphi_0 \in H^1_*(R^n)$ (which is defined in the obvious way). Multiplying (4.1) by φ_ε and integrating by parts, we obtain

$$p \int_{\Omega_\varepsilon} u_\varepsilon^{p-1} \varphi_\varepsilon^2 = 1.$$

Thus by (3.3) and (4.2) we have $p \int_{R^n} w^{p-1}(x - x_0)\, \varphi_0^2(x)\, dx = 1$ and hence $\varphi_0 \not\equiv 0$. Obviously φ_0 is a weak solution and hence classical solution of

$$\Delta \varphi_0(x) - \varphi_0(x) + p w^{p-1}(x - x_0)\, \varphi_0(x) = 0, \qquad x \in R^n.$$

By Lemma 2.6 (or Lemma 4.2 of [9]), we have that $\varphi_0(x)$ is a linear combination of $\partial w(x - x_0)/\partial x_i$, $1 \leqslant i \leqslant n$, i.e.,

$$\varphi_0(x) = c_1 \frac{\partial w(x - x_0)}{\partial x_1} + \cdots + c_n \frac{\partial w(x - x_0)}{\partial x_n} \tag{4.3}$$

for some contants c_i, $i = 1, ..., n$. Since $u_\varepsilon \in H^1_*(\Omega_\varepsilon)$, by (3.3), we have $w(x - x_0) \in H^1_*(R^n)$. Therefore x_0 is on the x_1-axis. Since $\varphi_0 \in H^1_*(R^n)$, $\partial\varphi_0/\partial x_i|_{x = x_0} = 0$, $2 \leqslant i \leqslant n$. On the other hand, it is easy to check that $\partial^2 w(0)/\partial x_i \partial x_j = 0$, $i \neq j$, and $\partial^2 w(0)/\partial x_i^2 \neq 0$.

Thus by (4.3), we have $c_2 = c_3 = \cdots = c_n = 0$ and hence $\varphi_0(x) = c_1(\partial w(x - x_0)/\partial x_1)$. Let $\alpha = \arccos(x_1 - x_{0,1}/|x - x_0|)$ ($x_{0,1}$ is the first component of x_0), $\rho = |x - x_0|$, $\theta = \arccos(x_1/|x|)$. Then

$$\begin{aligned} \frac{\partial}{\partial \theta}\left(\frac{\partial w(x - x_0)}{\partial x_1}\right) &= \frac{\partial}{\partial \theta}(w'(\rho)\cos(\alpha)) \\ &= \frac{\partial \alpha}{\partial \theta}\frac{\partial}{\partial \alpha}(w'(\rho)\cos(\alpha)) \\ &= \frac{\partial \alpha}{\partial \theta} w'(\rho)\sin(\alpha) \end{aligned}$$

But for points x on a sphere centered at the origin with radius less $|x_0|$, $\partial\alpha/\partial\theta$ changes sign, while $w'(\rho)\sin(\alpha)$ does not. Therefore $\varphi_0(x)$ cannot be in $H^1_*(R^n)$. A contradiction! ∎

Remark. In the above arguments, the condition $1 < p < 1 + 4/n$ is never used. But this condition is necessary in Lemma 3.4 of [2] which also holds in our situation, as can be easily checked.

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